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A proof that tidal heating in a synchronous rotation is always larger than in an asymptotic nonsynchronous rotation state

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Running head: Tidal heating at arbitrary eccentricity and obliquity

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Abstract:

In a recent paper, Wisdom (2007, Icarus, in press) derived concise expressions for the rate of tidal dissipation in a synchronously rotating body for arbitrary orbital eccentricity and obliquity. He provided numerical evidence that the derived rate is always larger than in an asymptotic non-synchronous rotation state at any obliquity and eccentricity. Here, I present a simple mathematical proof of this conclusion and show that this result still holds for any spin-orbit resonance.

Key words: planets: extrasolar, satellites; satellites, dynamics; orbital; tides, solid body

1. Introduction

Tidal heating is a source of energy which can have a strong influence on the thermal and internal history of celestial bodies. For solid bodies like the terrestrial planets or rocky satellites, local heating is presumed to arise from the conversion of the mechanical strain energy associated with time-dependent tidal distortion. This may occur when a rotating satellite librates on an eccentric orbit or has a non-zero obliquity.

For a homogeneous and incompressible synchronously rotating satellite, the expression of tidal heating as a function of the eccentricity has been calculated in detail by Peale and Cassen (1978). The derivation is made by calculating the power dissipated by the tide-raising force on each internal displaced constituent of the satellite. More recently, Wisdom (2004) generalized this expression to the second order in obliquity.

For giant gaseous planets that are not expected to be trapped in spin-orbit resonances but rather to reach an asymptotic nonsynchronous state, Levrard et al.(2007) argued that previous expressions are inadequate in this situation and derived new expressions of tidal dissipation for an asymptotic nonsynchronous rotation valid at arbitrary eccentricity and obliquity. Unfortunately, they compared their results to the expressions given in Wisdom (2004) and concluded that the rate of tidal heating in synchronous rotation is always lower than in an asymptotic nonsynchronous state. Using new derivations and useful formulae for the rate of tidal heating in a synchronously rotating body valid for any eccentricity and obliquity, Wisdom (2007) found the opposite result.

In this very short note, I provide a mathematical proof of this conclusion. Because solid (exo)planets and satellites are also expected eventually to despin to a more general state of spin-orbit resonance where the orbital period is some integer or half-integer times the rotation period (e.g. Goldreich and Peale 1966; Dobrovolskis 2007), I generalize this result to all the other spin-orbit resonances.

2. Comparison of tidal dissipation between a synchronously and asymptotically nonsynchronous rotating body

We consider the gravitational tides raised by an host planet on a satellite (the demonstration also holds for a planet around its central star). We use the simplest model of tidal response, generally called “viscous” model as described in Mignard (1980). He assumed a constant time lag for any frequency component of the tidal perturbation. In other words, the tidally deformed surface of the satellite always assumes the equipotential surface it would have formed a constant time lag Δt ago, in the absence of dissipation. In this case, the ratio $1/Q$ where Q is the satellite effective tidal dissipation factor is proportional to the frequency of the tides.

Here, I calculate the rate of tidal dissipation from the variation of the mechanical energy (rotational + orbital) of the satellite caused by tidally-driven perturbations in the satellite’s rotational

and orbital parameters (e.g. Hut 1981). In that case, the total energy is

$$E = \frac{1}{2}C\omega^2 - \frac{GM_s M_p}{2a}, \quad (1)$$

where ω is the satellite's rotation rate, C is its polar moment of inertia, M_p the mass of the host planet (primary body), M_s the satellite mass, and a is the orbit semimajor axis.

For a satellite locked into a synchronous resonance (1:1), it is necessary to add to the right hand side of equation (1), the external gravitational potential of the deformed satellite caused by its permanent quadrupole moment. Averaged over an orbital period, it is classically given by ¹

$$V = -\frac{3}{4}H(1, e)(B - A)n^2 \cos 2\gamma \quad (2)$$

where $H(1, e)$ is the Hansen's coefficient for the synchronous resonance, e is the orbital eccentricity, n is the orbital mean motion, A and B are the satellite's equatorial moments of inertia and γ is the resonant angle with

$$d\gamma/dt = \omega - n. \quad (3)$$

The gravitational restoring torque exerted by the planet on the quadrupole moment maintains the spin in the resonance.

Combining equations (1),(2) and (3), the rate of tidal dissipation within the satellite then is

$$\dot{E}_{tidal} = -\frac{dE}{dt} = -C\omega \times \frac{d\omega}{dt} - \frac{3}{2}H(1, e)(B - A)n^2 \frac{d\gamma}{dt} \sin 2\gamma - \frac{GM_s M_p}{2a^2} \times \frac{da}{dt}. \quad (4)$$

Using the equation for the rotational motion of the satellite averaged over an orbital period (e.g. Murray & Dermott 1999) :

$$C \frac{d^2\gamma}{dt^2} = C \frac{d\omega}{dt} = -\frac{3}{2}H(1, e)(B - A)n^2 \sin 2\gamma + \Gamma_{tidal} \quad (5)$$

where Γ_{tidal} is the mean tidal torque acting to brake the spin of the satellite, equation (4) becomes

$$\dot{E}_{tidal} = -\omega \Gamma_{tidal} + \frac{3}{2}H(1, e)(B - A)n^3 \sin 2\gamma - \frac{GM_s M_p}{2a^2} \times \frac{da}{dt}. \quad (6)$$

For most of satellites and planets of the solar system, the libration period is much longer than the orbital period but much shorter than the typical despinning timescale so that $\langle \sin(2\gamma) \rangle = 0$ over a libration period. Averaging the equation (6) over a libration period or over secular timescales longer than the libration period then leads to

$$\langle \dot{E}_{tidal} \rangle = -\omega \langle \Gamma_{tidal} \rangle - \frac{GM_s M_p}{2a^2} \times \left\langle \frac{da}{dt} \right\rangle. \quad (7)$$

¹For simplicity, a zero obliquity is assumed but the same conclusion holds at any obliquity.

For the “viscous” tidal model, the average tidal torque is given by (e.g. Hut 1981; Néron de Surgy & Laskar 1997):

$$\langle \Gamma_{tidal} \rangle = -\frac{K}{n} \left[(1+x^2) \Omega(e) \frac{\omega}{n} - 2x N(e) \right] \quad (8)$$

where ε is the satellite’s obliquity (the angle between the satellite equatorial and orbital planes), $x = \cos \varepsilon$ and

$$K = \frac{3}{2} \frac{k_2}{Q_n} \left(\frac{GM_s^2}{R_s} \right) \left(\frac{M_p}{M_s} \right)^2 \left(\frac{R_s}{a} \right)^6 n \quad (9)$$

where k_2 is the potential Love number of degree 2, R_s is the satellite radius and $Q_n = (n \Delta t)^{-1}$ is the annual tidal quality factor. The eccentricity-dependent functions $N(e)$ and $\Omega(e)$ are then

$$\Omega(e) = \frac{1 + 3e^2 + \frac{3}{8}e^4}{(1 - e^2)^{9/2}}$$

and

$$N(e) = \frac{1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6}{(1 - e^2)^6}.$$

In the same way, we have (e.g. Hut 1981; Néron de Surgy & Laskar 1997):

$$\left\langle \frac{da}{dt} \right\rangle = 4a^2 \left(\frac{K}{GM_s M_p} \right) \left[N(e) x \frac{\omega}{n} - N_a(e) \right], \quad (10)$$

where

$$N_a(e) = \frac{1 + \frac{31}{2}e^2 + \frac{255}{8}e^4 + \frac{185}{16}e^6 + \frac{25}{64}e^8}{(1 - e^2)^{15/2}}.$$

Substituting equations (8) and (10) into the equation (7) provides the average rate of energy dissipation

$$\left\langle \dot{E}_{tidal} \right\rangle = 2K \left[N_a(e) + \frac{1+x^2}{2} \Omega(e) - 2xN(e) \right] \quad (11)$$

using $\omega \simeq n$ and valid to any order in eccentricity and obliquity.

We checked that this expression fully agrees with the equation (30) of Wisdom (2007) calculated for a homogeneous, incompressible and small and/or rigid enough body that the radial displacement Love number h_2 is $5k_2/3$. Note that our derivation does not require such an hypothesis and all the uncertainties in the radial distribution of material and its physical properties (e.g. density, compressibility, elasticity) are lumped into the k_2 parameter. For a homogeneous and incompressible body, the Love number of degree 2 is given by the well-known formula $k_2 = (3/2)/(1 + 19\mu/(2\rho g R_s))$ where ρ is the density and μ is the elastic shear modulus. For real rocky material, the effect of compressibility can not be actually neglected because μ is close to λ where λ is classically the Lamé’s parameter which is a measure of compressibility ($\lambda \rightarrow \infty$ for an incompressible material). However, if deviations from incompressibility and homogeneous density requires only small corrections, a varying elastic shear modulus could have important consequences (e.g. Peale and Cassen 1978). For a homogeneous and incompressible fluid planet, $k_2 = 3/2$ but is

about one order of magnitude smaller for realistic profiles of density. Furthermore, when the strong effect of compressibility is taken account, it is worthy to note that Love numbers could experience dramatic and unexpected variations (e.g. Hurford et al. 2002).

Let us now consider the same satellite (or planet) that is not locked into a synchronous resonance. This may occur because the body is essentially fluid or near-fluid and does not have a permanent quadrupole moment, or because its eccentricity is too large to allow a capture into a synchronous resonance (e.g. Goldreich and Peale 1966).

Because the satellite’s orbital angular momentum is generally much larger than its spin angular momentum, gravitational tides affect the spin’s properties (rotation, obliquity) much faster than the orbit’s properties. For the “viscous” model, tides ultimately reduce the obliquity to zero on the same time scale as the despinning (e.g. Hut 1981). Here, we consider that other separate mechanisms may maintain or excite the obliquity to a non-zero value (due for exemple to a capture in a Cassini state, to perturbations by a companion, ...). Setting $\langle \Gamma_{tidal} \rangle = 0$ in the equation (8), the spin evolves to its asymptotic equilibrium rate of rotation

$$\omega_{eq} = \frac{N(e)}{\Omega(e)} \frac{2x}{1+x^2} n \quad (12)$$

while the eccentricity and the obliquity are assumed to vary more slowly. Note that a non-zero eccentricity favors a supersynchronous rotation (that is, a rotational period shorter than the orbital period) while a non-zero obliquity favors a subsynchronous rotation. As a consequence, the asymptotic rate of rotation can be higher or lower than the synchronous rotation rate depending on the eccentricity and obliquity values.

Once the spin reaches its “pseudo-equilibrium”, tidal energy is then dissipated within the satellite only at the expense of the orbital energy. From equation (7), it comes

$$\langle \dot{E}_{tidal} \rangle = -(GM_p M_s)/(2a^2) \times \langle da/dt \rangle.$$

The derived rate of tidal dissipation is then (see Levrard et al. 2007)

$$\langle \dot{E}_{tidal} \rangle = 2K \left[N_a(e) - \frac{N^2(e)}{\Omega(e)} \frac{2x^2}{1+x^2} \right]. \quad (13)$$

It is now easy to compare the rate of tidal dissipation in a synchronously rotating body (Equation (11)) with that in an asymptotic nonsynchronous rotation rate (Equation (13)) for any eccentricity and obliquity. The former is larger than the latter if

$$\frac{1+x^2}{2} \Omega(e) - 2xN(e) \geq -\frac{N^2(e)}{\Omega(e)} \frac{2x^2}{1+x^2}. \quad (14)$$

We found that this condition is always verified because the previous equation is equivalent to

$$[(1+x^2)\Omega(e) - 2xN(e)]^2 \geq 0. \quad (15)$$

As a consequence, the dissipation rate in synchronous rotation is always equal to or larger than that in asymptotic rotation for any obliquity and eccentricity. Note that the equality holds when the asymptotic rate of rotation is synchronous, that is the satellite's eccentricity and obliquity verify

$$\frac{N(e)}{\Omega(e)} \frac{2x}{1+x^2} = 1. \quad (16)$$

For a zero obliquity, it is possible only if the eccentricity is zero. The function $N(e)/\Omega(e)$ is monotonic and always increases with the eccentricity so that there is only one eccentricity and obliquity value that satisfies the previous condition.

2.1. Generalization to other spin-orbit resonances

Let us now assume that the satellite is locked into another spin-orbit resonance. We have $\omega \sim pn$ where p is an integer or a half-integer. Using equations (7), (8) and (10), the rate of tidal dissipation for a resonant rotation is then

$$\langle \dot{E}_{tidal} \rangle = 2K \left[N_a(e) + p^2 \frac{1+x^2}{2} \Omega(e) - 2pxN(e) \right]. \quad (17)$$

This rate of dissipation is larger than the asymptotic non-resonant rate of rotation if

$$p^2 \frac{1+x^2}{2} \Omega(e) - 2pxN(e) \geq -\frac{N^2(e)}{\Omega(e)} \frac{2x^2}{1+x^2}. \quad (18)$$

Once again, this condition is always verified because this relationship is also equivalent to

$$[p(1+x^2)\Omega(e) - 2xN(e)]^2 \geq 0. \quad (19)$$

The equality holds when the asymptotic rate of rotation is equal to the rotation rate in the spin-orbit resonance that is

$$\frac{\omega}{n} = \frac{N(e)}{\Omega(e)} \frac{2x}{1+x^2} = p. \quad (20)$$

Only one eccentricity and obliquity verify this equality.

3. Conclusion

We provide a mathematical demonstration that the rate of tidal dissipation in a synchronously rotating satellite is larger than that in a asymptotic nonsynchronous rotation state for any obliquity and eccentricity.

We also show that this property still holds for the other spin-orbit resonances. It would be interesting to investigate whether these results are also valid for other tidal models.

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